Restoring site percolation on damaged square lattices

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Restoring site percolation on a damaged square lattice with the nearest neighbor (N^2) is investigated using two different strategies. In the first one, a density y of new sites are created on the empty sites with longer range links, either next-nearest neighbor (N^3) or next-next-nearest neighbor (N^4), but without N^2. In the second one, new longer range links N^3 or N^4 are added to N^2 but only for a fraction v of the remaining nondestroyed sites. Starting at $p_c(N^2)$, with a density x of randomly destroyed sites, the values of y_c and v_c, which restore site percolation, are calculated for both strategies with, respectively, N^3 and N^4 using Monte Carlo simulations. Results are obtained for the whole range $0 \leqslant x \leqslant p_c(N^2)$.

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I. INTRODUCTION

For several decades, the calculation of percolation thresholds has been an ongoing technical challenge (see [1,2] for recent ones). Up to now, analytical solutions have been limited to few two-dimensional lattices. However, percolation thresholds have been estimated very accurately for an increasing large number of systems using computer Monte Carlo simulations and powerful computers [3,4]. High-temperature series expansion have been also extensively used [5].

The most studied case is the hypercube due to its simple computer representation with available numerical estimates of percolation threshold up to 13 dimensions [3,6,7]. Some universal formulas have also been suggested [2,8]. Percolation phenomenon [3,4] is also found to occur on a large variety of complex networks where sites are not distributed regularly [9–11]. The resistance to intentional or random attacks has been studied for these complex networks [10], but not for regular lattices.

In this paper we address this problem and investigate how to restore site percolation on a square lattice with nearest-neighbor links (N^2) once a random attack has occurred. We consider the square lattice at the percolation threshold $p_c \equiv p_c(N^2)$, once a fraction x of the initial p_c sites have been randomly destroyed. Two strategies are suggested. In the first, a density y of new sites are created on the empty sites with longer range bonds, either next-nearest neighbor (N^3) or next-next-nearest neighbor (N^4). It is worth noting that these additional sites with N^3 or N^4 links have no N^2 links.

In the second strategy, no additional sites are created but instead new longer range links, either N^3 or N^4, are added to the N^2, links but only for a fraction v of the remaining undamaged (p_c-x) sites. Accordingly, $v(p_c-x)$ sites have N^2 plus either N^3 or N^4 links, while $(1-v)(p_c-x)$ sites have only their initial N^2 links. Figure 1 shows sites with, respectively, N^2, N^3, N^4, (N^2+N^3), and (N^2+N^4) links on the lattice.

Given a fixed density of destroyed sites x, the associated values y_c and v_c, which restore site percolation, are calculated with, respectively, N^3 and N^4 using Monte Carlo simulations. Results are obtained for the whole range $0 \leqslant x \leqslant p_c$, which in turn leads to new site percolation thresholds, $\pi_3 \equiv (p_c-x+y_c)$ for N^3, $\pi_4 \equiv (p_c-x+y_c)$ for N^4, $\pi_{23} \equiv v_c(p_c-x)$ for N^3, and $\pi_{24} \equiv v_c(p_c-x)$ for N^4. The first two are obtained for the first strategy while the last two are for the second strategy.

However, to allow a more simple evaluation of above expressions we reformulate the problem as follows. Since for each strategy we have two kinds of sites with respect to their links, we consider instead of the above problem a square lattice with a density π of occupied sites, of which a given fraction q has one kind of link, the initial N^2, while remaining fraction $(1-q)$ have the other kind, N^3 or N^4 for the first strategy and (N^2+N^3) or (N^2+N^4) for the second one.

Then, given the mixing neighborhood parameter q, we evaluate the threshold π_c using Monte Carlo simulations. From π_c we can then go back to our former problem and extract the values of the pairs $\{x,y_c\}$ and $\{x,v_c\}$. For the first strategy $x=p_c-q\pi_c$ and $y_c=(1-q)\pi_c$, while for the second $x=p_c-\pi_c$ and $v_c=(1-q)$. Monte Carlo simulations are run over the whole range $0 \leqslant q \leqslant 1$. Associated thresholds are presented in Fig. 2.

II. CALCULATION

To perform our calculations of percolation thresholds, we are using the Hoshen–Kopelman algorithm (HKA) [12] from existing computational techniques [13–15]. With HKA each occupied site gets a label. The sites in the same cluster have the same labels and different labels are assigned to different

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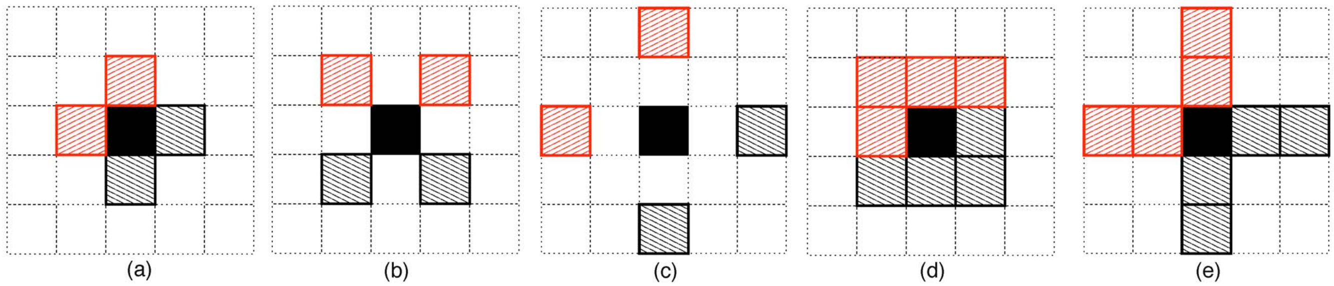

FIG. 1. (Color online) Various site neighborhoods on the square lattice: (a) N^2, (b) N^3, (c) N^4, and examples of their combinations: (d) N^2+N^3, (e) N^2+N^4.

clusters, thus allowing us to recognize which sites belong to which clusters. When different sites have different neighbors—as presented in Fig. 3—the situation becomes more complicated than for homogeneous case [3,16]. In Fig. 3, black site *is not* N^2 of dark (red) one, while the dark (red) one *is* N^3 of the black one. Note that one may recognize these two sites as parts of the same cluster or not, depending on sites' inspector intentions. To avoid such ambiguity we classify them as the members of one common cluster when we go in type-writer order, i.e., from top-to-bottom and from left-to-right, but we leave them separated when we go in reverse-type-writer order, i.e., from bottom-to-top and from right-to-left. However, as the lattice and sites distribution is homogeneous one can meet both these situations equally often and particular chose of any sites inspection order does not influence the results. Technically, it is realized by considering each site as having effectively only slashed (red) "half" of their neighborhood presented in Fig. 1. This technical trick is used only to simplify the computational labeling of sites and has no effect at all on the physical results.

III. RESULTS

Technically the percolation thresholds values π_c are evaluated from the crossing point of two different curves showing both the dependence of the percolation probability P versus the sites occupation probability p for two different linear sizes $L=100$ and 500 (see Fig. 4). The results are averaged over $N_{run}=10^3$ and 10^4 for $L=500$ and 100, respectively. They are presented in Fig. 2. As a matter of fact, each site has only "half" of their neighborhood as we have explained earlier.

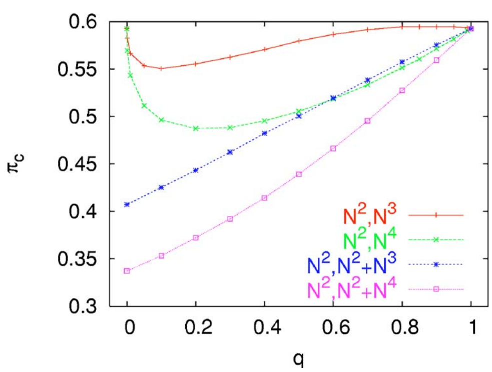

FIG. 2. (Color online) The percolation threshold π_c dependence on the neighborhoods mixing parameter q. The lines are guides for the eyes.

For the first strategy—where new type of sites are added with long-range links N^3 or N^4—the extracted corresponding values of the pairs $\{x, y_c\}$ are collected in Tables I and II and presented in Fig. 5(a).

For the second strategy—where additional longer-range links N^3 or N^4 are added to a fraction v of the formerly nondestroyed sites—the results are collected in Table III and presented in Fig. 5(b). The extracted corresponding values of the pairs $\{x, v_c\}$ are included with x_{23} and x_{24} referring to, respectively, N^3 and N^4 links.

IV. DISCUSSION

Homogeneous cases $q=0$ and $q=1$ were indeed studied recently in the context of a systematic increase of the range of links in Ref. [18]. Similar checks for bi-colored lattices with N^2 and N^3 neighborhoods were also performed in Ref. [19].

We note that for the site reconstruction process, at $x \approx 0$ or $x \approx p_c$, the difference between originally used N^2 sites and their N^3 and/or N^4 substitutes vanishes.

• In the limit $x \to 0$, where damages are very scarce, the reconstruction process is not too costly independent of what kind of "bricks" (sites with N or N^4) is used.

• On the other hand, when $x \to p_c$, the system must be completely reconstructed starting from almost zero. Since all of homogeneous N^2, N^3, and N^4 lattices have the same per-

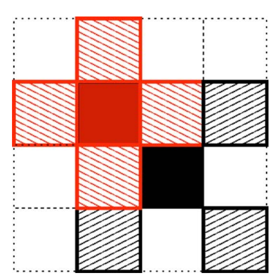

FIG. 3. (Color online) Are full-filled black and dark (red) sites in one cluster? The dark (red) site has N^2 while the black one has N^3. Our answer may be positive and the HKA will give such an answer only with the probability of 50%. The answer is sites labeling order dependent.

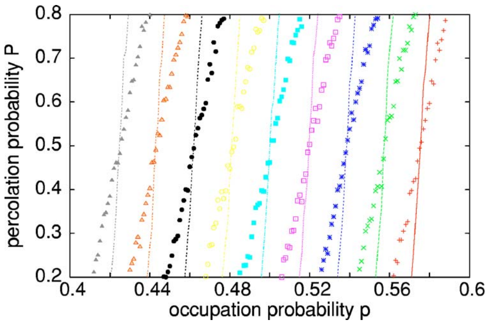

FIG. 4. (Color online) The percolation probability P dependence on the occupation probability p for different values of mixing parameter q which changes from 10% to 90% every 10% from right to left. The N^2 and (N^2+N^3) neighborhoods are mixed. The symbols correspond to $L=100$, while lines to $L=500$. The crossing points predict the percolation thresholds $\pi_{23}=\pi_c(q,N^2,N^2+N^3)$.

TABLE I. The fraction y_c of sites with N^3 which must be added to reconstruct the percolation phenomenon—which occurs at $\pi_3=\pi_c(q,N^2,N^3)$—when only (p_c-x) of sites with N^2 is occupied.

$1-q$	x	y_c	π_3
0.0	0.0000	0.0000	0.592
0.1	0.0574	0.0594	0.594
0.3	0.1783	0.1773	0.591
0.4	0.2404	0.2344	0.586
0.5	0.3025	0.2895	0.579
0.6	0.3640	0.3420	0.570
0.8	0.4810	0.4440	0.555
0.95	0.5645	0.5254	0.553
0.99	0.5863	0.5603	0.566
1.0	0.5920	0.5920	0.592

TABLE II. The fraction y_c of sites with N^4 which must be added to reconstruct the percolation phenomenon—which occurs at $\pi_4=\pi_c(q,N^2,N^4)$—when only (p_c-x) of sites with N^2 is occupied.

$1-q$	x	y_c	π_4
0.0	0.0000	0.0000	0.592
0.05	0.0401	0.0291	0.581
0.2	0.1512	0.1102	0.551
0.3	0.2189	0.1599	0.533
0.4	0.2812	0.2072	0.518
0.5	0.3395	0.2525	0.505
0.7	0.4456	0.3416	0.488
0.9	0.5424	0.4464	0.496
0.99	0.5866	0.5376	0.543
1.0	0.5920	0.5920	0.592

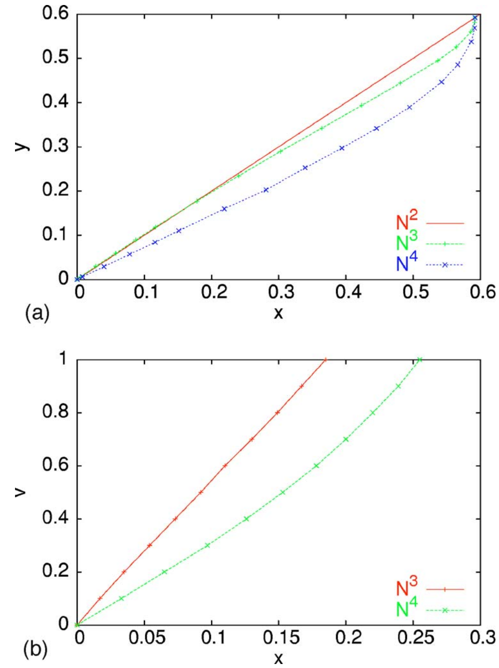

FIG. 5. (Color online) The fraction of sites which must be (a) reoccupied (y) or (b) enriched with long-range links (v) to recover the percolation phenomenon when the fraction x of sites with the N^2 was emptied. The lines are guides for the eyes.

colation threshold $p_c(N^2)=p_c(N^3)=p_c(N^4)=0.592\,746\,0$ [17,18] the cost of reaching a new percolation threshold becomes again neighborhood independent.

• Note, that in both cases (N^3 and N^4), there is an optimal ratio $q/(1-q)$ for which the percolation threshold π_c is the lowest.

• For N^3 sites used for the reconstruction process $\pi_3>p_c$, when $0.8<q<1$, i.e., when damages are relatively small ($x<0.2$). In that case addition sites with diagonal bonds do not help much with restoring the percolation, but increase total sites density.

TABLE III. The fraction v_c of sites with N^2 which must be enriched with N^3 or N^4 bonds to reconstruct the percolation phenomenon—which occurs at, respectively, π_{23} or π_{24}—when only (p_c-x) of sites with N^2 were saved after attack; x_{23} and x_{24} refer to x for, respectively, N^3 and N^4 links.

$1-q$	x_{23}	π_{23}	x_{24}	π_{24}	v_c
0.0	0.000	0.592	0.000	0.592	0.0
0.1	0.017	0.575	0.033	0.559	0.1
0.2	0.035	0.557	0.065	0.527	0.2
0.3	0.054	0.538	0.097	0.495	0.3
0.4	0.073	0.519	0.126	0.466	0.4
0.5	0.092	0.500	0.153	0.439	0.5
0.6	0.110	0.482	0.178	0.414	0.6
0.7	0.130	0.462	0.200	0.392	0.7
0.8	0.149	0.443	0.220	0.372	0.8
0.9	0.167	0.425	0.239	0.353	0.9
1.0	0.185	0.407	0.255	0.337	1.0

For the second scenario, we found a critical density of sites x_c above which site destruction is not compensable by adding long-range bonds, even to all of remaining sites. For $x > x_c$ the site concentration falls below the threshold p_c for the square lattice with all sites having the same mixed neighborhood, for instance, N^2+N^3 or N^2+N^4 [18]. The critical values $x_c(N)=\pi_c(0,N^2,N^2+N)-p_c$ are 0.185 and 0.255 for $N=N^3$ and N^4, respectively.

For both reconstruction strategies, the percolation is restored more easily if neighborhoods with larger links are employed as presented in Fig. 5.

V. CONCLUSIONS

To conclude, in contrast to Ref. [10], which focused on how to destroy connectivity on a given lattice, we concentrate on effective reconstruction after a random destruction. Using Monte Carlo simulations we have reported square lattice site percolation thresholds π_c for given neighborhoods characterized by a mixing parameter q and various pairs of mixed neighborhoods built from basic ones, i.e., N^2, N^3, and N^4. We have showed quantitatively that restoring requires less sites to be created when longer links are employed for site reconstruction process at intermediate range of damages. The strategy involving bond enrichment fails if damages are too large. The critical damages size x_c depends on the percolation threshold $p_c(N^2+N)$, where N is the kind of bonds used in the enriching process.

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